Uitwerking midtoets 04-06-2009
Lineaire Algebra
Collegejaar 2008/2009

Door dr. ir. Roel Luppes tijdens het college van 18-06-2009

27 juni 2009

Opdracht 1

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & \alpha \end{bmatrix} \vec{b} = \begin{bmatrix} 1 \\ 0 \\ \beta \end{bmatrix}$$

a)

α = -1, β = 5 A$\vec{x} = \vec{b}$ oplossen.

$$\begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 5 \\ 0 & -1 & -1 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 & -9 \\ 1 & 0 & 0 & 5 \\ 0 & -1 & -1 & 5 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 0 & 1 & 0 & -9 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & -1 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 & -9 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

y = -9
x = 5
z = 4

$$\vec{x} = \begin{bmatrix} 5 \\ -9 \\ 4 \end{bmatrix}$$

b)

A$\vec{x} = \vec{b}$ precies 1 oplossing?

det(A) $\neq 0 \rightarrow 1$ oplossing.

$$det(A) = 2\begin{vmatrix} 1 & 1 \\ -1 & \alpha \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & \alpha \end{vmatrix} + 0 \cdot Bla...$$

$=$ 2(α + 1) - 1(α - 0) $=$ 2α + 2 - α $=$ α + 2 $\neq 0 \Rightarrow$ Als $\alpha \neq$ -2 precies 1 oplossing.

c)

Geen oplossing?
α = -2

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & \beta \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 0 & -1 & -2 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & \beta \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 0 & -1 & -2 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & \beta - 1 \end{array}\right) \rightarrow 0 = \beta - 1$$

Geen oplossing als $\beta \neq 1$.

Opgave 2

$$\vec{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \vec{b} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \vec{c} = \begin{pmatrix} \alpha \\ -4 \\ 3 \end{pmatrix}$$

a)

Hoek tussen $\vec{a}$ en $\vec{b}$.
$\cos(\alpha) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} = \frac{2+0+0}{\sqrt{2} \cdot \sqrt{8}} = \frac{2}{\sqrt{16}} = \frac{1}{2}$
$\rightarrow \alpha = \frac{\pi}{3}$

b)

<u>vector $\perp \vec{a}$ en $\perp \vec{b}$</u>

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0 \rightarrow x + z = 0$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = 0 \rightarrow 2x + 2y = 0$$

$$\begin{matrix} z = -x \\ y = -x \end{matrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -x \\ -x \end{pmatrix} = x \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

c)

Wanneer c $\in$ span$\{\vec{a}, \vec{b}\}$?

$$\begin{pmatrix} \alpha \\ -4 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \rightarrow \left.\begin{matrix} \alpha = c_1 + 2c_2 \\ -4 = 2c_2 \\ 3 = c_1 \end{matrix}\right\} \begin{matrix} c_1 = 3 \\ c_2 = -2 \end{matrix}$$

α = 3 + 2 ·(-2) = -1
α = -1 alles ok.
c $\in$ span$\{\vec{a}, \vec{b}\}$.

d)

Wanneer zijn $\vec{a}$, $\vec{b}$ en $\vec{c}$ lineair afhankelijk?
Opgave c) als α = -1, $\vec{c} = c_1 \cdot \vec{a} + c_2 \cdot \vec{b}$ lineair onafhankelijk.
Lineair afhankelijk als $\alpha \neq$ -1.
Opm. det.

$$\begin{pmatrix} \alpha & 1 & 2 \\ -4 & 0 & 2 \\ 3 & 1 & 0 \end{pmatrix} \neq 0$$

Ogave 3

$$A = \begin{pmatrix} 2 & -6 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 2 & -1 \end{pmatrix}, A_{11} = \begin{pmatrix} 2 & -6 \\ 1 & 3 \end{pmatrix}, A_{22} = \begin{pmatrix} 3 \end{pmatrix}, A_{33} = \begin{pmatrix} 3 & -1 \\ 2 & -1 \end{pmatrix}$$

a)

A^{-1} ?

$$A^{-1} = \begin{pmatrix} A_{11}^{-1} & & \\ & A_{22}^{-1} & \\ & & A_{33}^{-1} \end{pmatrix}$$

$$A_{33}^{-1} = \begin{pmatrix} 3 & -1 \\ 2 & -1 \end{pmatrix}^{-1} = \frac{1}{-1} \begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix}$$

$A_{22}^{-1} = \frac{1}{3}$

$$A^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{12} & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 & -3 \end{pmatrix}$$

b)

$$det(A) = det \begin{pmatrix} 2 & -6 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & -1 \\ \alpha & 0 & 0 & 2 & -1 \end{pmatrix}$$

Blok L matrix $\rightarrow$

$$det \begin{pmatrix} 2 & -6 \\ 1 & 3 \end{pmatrix} \cdot det(3) \cdot det \begin{pmatrix} 3 & -1 \\ 2 & -1 \end{pmatrix}$$

= (6 + 6)·(3)·(-3 + 2) = 12 · 3·(-1) = -36

Opgave 4

$$A = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 2 & 2 \\ 0 & -1 & -1 \end{pmatrix}$$

$$\vec{P} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \vec{Q} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \vec{R} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

$\mathrm{T}(\vec{x}) = \mathrm{A}\vec{x}$, $\vec{x} \to \mathrm{A}\vec{x}$.

a)

Beeld van $\vec{P}$?

$$A\vec{p} = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 2 & 2 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \\ -5 \end{pmatrix}$$

b)

Origineel van $\vec{q}$?
Stel

$$\vec{q} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ -1 & 2 & 2 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \to \left(\begin{array}{ccc|c} 2 & 1 & 0 & 2 \\ -1 & 2 & 2 & -1 \\ 0 & -1 & -1 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 1 & 0 & 2 \\ -1 & 0 & 0 & -3 \\ 0 & -1 & -1 & -1 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & -4 \\ -1 & 0 & 0 & -3 \\ 0 & -1 & -1 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 0 & 1 & 0 & -4 \\ -1 & 0 & 0 & -3 \\ 0 & 0 & -1 & -5 \end{array}\right) \to \left(\begin{array}{ccc|c} 0 & 1 & 0 & -4 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right) \begin{array}{r} y = -4 \\ x = 3 \\ z = 5 \end{array}$$

Dit is het enige origineel, want er zijn geen conflicten (geen afhanklijkheden).

$$\begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$$

c)

S = T^{-1}, S(x) = A^{-1}x.
Beeld van $\vec{R}$ onder afbeelding S?
$\vec{R} \rightarrow$ S($\vec{R}$) = $\vec{y} = A^{-1}\vec{R}$
A$\vec{y}$ = A, $A^{-1}\vec{R}$ = I$\vec{R}$
A$\vec{y}$ = $\vec{R}$

$$\begin{pmatrix} 2 & 1 & 0 \\ -1 & 2 & 2 \\ 0 & -1 & -1 \end{pmatrix}, \vec{y} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \rightarrow \vec{y} = \vec{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Opmerking: vegen mag ook.

Opgave 5

$$M = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 2 \end{pmatrix}, \vec{R_0} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$\vec{R}_{k+1}$ = M$\vec{R}_k$

a)

Eigenwaarden van M? ($\lambda = 1$ gegeven)

$$det \begin{pmatrix} 2-\lambda & 1 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & -1 & 2-\lambda \end{pmatrix} = (2-\lambda) \cdot \begin{vmatrix} 1-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 0 \\ -1 & 2-\lambda \end{vmatrix}$$

= (2 - λ)·((1 - λ)·(2 - λ) + 1) - 1·(2 - λ)
= (2 - λ)·((1 - λ)·(2 - λ) + 1) + (2 - λ)·(-1)
= (2 - λ)·((1 - λ)·(2 - λ) + 1 - 1)
= (2 - λ)·(1 - λ)·(2 - λ)

$\lambda_1 = 1$
$\lambda_{2,3} = 2$

b)

$\lambda = 1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right)$$

x = -z
y = z

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ z \\ z \end{pmatrix} = z \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$\lambda = 2$

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) \begin{array}{l} \rightarrow x = -y \\ \rightarrow \; y = 0 \\ \rightarrow x = -7 \end{array} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

c)

Diagonalisatie? → Nee, maar 2 eigenwaarden.

d)

Wat is $\vec{R}_{100}$? $\vec{R}_{100} = M^{100} R_0$

$$Bij \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$M\vec{R}_0 = 1 \cdot \vec{R}_0 = \vec{R}_0$
$M^{100} \cdot \vec{R}_0 = \vec{R}_0$
dus $\vec{R}_{100} = \vec{R}_0$